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# *THEORETICAL NOTE*

# Parallel Psychometric Functions from a Set of Independent Detectors

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Consider combining several elementary detectors by the extreme decision rule of responding "no" only when all elementary detectors respond "no" and "yes" otherwise. The question raised is: Which psychometric functions for the detectors have the property that the resulting psychometric function is simply the original function displaced in the logarithm of the physical scale? The answer is $p(I) = 1 - e^{-\alpha I^{\beta}}$, $\alpha$, $\beta > 0$.

Detection data are often assumed to arise by combining the decisions of a number of more elementary detectors. One extreme assumption is that the elementary detectors are binary and the final decision is affirmative if and only if at least one detector reports a detection. Obviously a number of other decision rules are possible—one might, for example, require some fraction of the elementary detectors to report detection. Such rules are more flexible and may prove more prudent or realistic in particular detection situations. Smith and Wilson (1953) have analyzed such systems in some detail. The appeal of the extreme rule is its simplicity, and so it is not surprising that discussions of this extreme rule occur in many different areas as a kind of null model. In certain classical psychophysical experiments, for example, the influence of signal duration or area on detection can be treated in this way, as can complex signals composed of one or more separate parts.

The psychometric functions observed in such experiments exhibit a surprisingly simple relation. Consider, for example, two different signal durations. Presumably the shorter duration involves fewer elementary detectors than does the larger one, but when one measures the psychometric function at each duration as a function of log intensity, one usually finds that both psychometric functions have essentially the same shape. That is to say, these psychometric functions are nearly parallel, the one simply being the other displaced to the left or right by an amount that is a function of the parameter value being varied—in this example, signal duration.

Such results were interpreted by Green, McKey, and Licklider (1959) as unfavorable to the putative model for the following reason. Let $p$ denote the elementary detection probability, and let $p_n$ denote that of a system involving $n$ independent elementary detectors. Assuming the decision rule stated, it is obvious that

$$p_n = 1 - (1 - p)^n. \qquad (1)$$

Since the relation between $p_n$ and $p$ is nonlinear, it was argued that the shape of the psychometric function must change as $n$ varies. Certainly this is true for a variety of simple psychometric functions, and if $n$ is large enough, this change should be easily noticed, even in empirical data.

However, as Quick (in press) has pointed out, the function

$$p(I) = 1 - e^{-\alpha I^{\beta}}, \quad \alpha > 0, \quad \beta > 0 \qquad (2)$$

has the property that $p_n$ in Equation 1 is of the same form provided $I$ is multiplicatively displaced by a function of $n$. Thus, on a logarithmic scale of intensity, $p(I)$ and $p_n(I)$ are parallel. The argument is simple:

$$\begin{aligned} p_n &= 1 - (1 - p)^n \\ &= 1 - (e^{-\alpha I^{\beta}})^n \\ &= 1 - e^{-\alpha (n^{1/\beta} I)^{\beta}}. \end{aligned}$$

So the function given in Equation 2 has the

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property that has been observed empirically. The main purpose of this note is to prove that it is the only function with the property that for some function $a$ on the positive integers and some cumulative probability function $p$,

$$p_n(I) = p[a(n)I]. \tag{3}$$

Before deriving this result and commenting on it, we digress briefly on two points. The first is the issue of false alarms. If we denote by $\epsilon$ the false-alarm rate of an elementary detector, then by Equation 1, the false-alarm rate of the system is $1 - (1 - \epsilon)^n$ which, when $\epsilon$ is small, is approximately $n\epsilon$. It is clear that if the system has a large number of detectors, its false-alarm rate will be modest only if the false-alarm value of each detector is extremely low. This is one reason that we view the model as an extreme one.

The second point to observe is that Equation 2 has appeared in the detection literature at least as early as Brindley (1963). He assumed that $I$ is proportional to the number of quanta impinging on an elementary detector, that there are $n$ independent detectors, and that $m$ quanta are required to excite any one of them. He claimed to show—though we believe the argument wrong (see Appendix)—that for large $n$ the process approaches that of Equation 2 with $\beta = m$ and $\alpha = n/m$!

The same mathematical problem has been treated in a wholly different literature, namely, that concerned with the distribution of the largest of $n$ identically distributed random variables. There it is shown that Equation 2 is one of three possible forms for the asymptotic distribution. The key paper is Fisher and Tippett (1928); it is summarized in Gumbel (1958) and Feller (1966). Feller's discussion is considerably more careful concerning the conditions under which the solution to the corresponding extremal problem is unique. Yellott, who has been working on a related problem in choice theory (Note 1), brought this literature to our attention and showed us how to use Feller's results.[1]

## *Form of the Psychometric Function*

THEOREM. *Assume that the psychometric function* p *is a (cumulative) distribution function on the nonnegative reals (i.e., it is increasing and onto [0, 1]) and that Equations 1 and 3 hold, then*

$$p(I) = 1 - e^{-\alpha I^\beta}, \quad \alpha > 0, \quad \beta > 0. \tag{2}$$

*Proof.* From Equations 1 and 3,

$$[1 - p(I)]^n = 1 - p[a(n)I].$$

Let

$$g(x) = 1 - p(1/x), \tag{4}$$

then it is easy to see that $g$ is a distribution function, that $g^n$ is the distribution of the largest of $n$ independent random variables, each with the distribution function $g$, and that

$$\begin{aligned} g(x)^n &= [1 - p(1/x)]^n && \text{(Equation 4)} \\ &= 1 - p_n(1/x) && \text{(Equation 1)} \\ &= 1 - p[a(n)/x] && \text{(Equation 3)} \\ &= g[x/a(n)]. && \text{(Equation 4)} \end{aligned}$$

So if $\mathbf{X}_n$ is a random variable with distribution $g^n$, then $\mathbf{X}_n/a(n)$ has the distribution

$$\begin{aligned} P\left[\frac{\mathbf{X}_n}{a(n)} \leq x\right] &= P[\mathbf{X}_n \leq a(n)x] \\ &= g[a(n)x]^n \\ &= g(x). \end{aligned}$$

According to Feller (1966, pp. 270–271), if the distributions of $\mathbf{X}_n/a(n)$ tend to a distribution $g$ not concentrated at 0—in this case, they all have the same distribution $g$—then

$$g(x) = e^{-\alpha x^\rho}, \quad \alpha > 0, \quad \rho < 0.$$

Thus,

$$\begin{aligned} p(I) &= 1 - g(1/I) \\ &= 1 - e^{-\alpha(1/I)^\rho} \\ &= 1 - e^{-\alpha I^\beta}, \quad \beta = -\rho > 0. \end{aligned}$$

## *Application*

If we perform an experiment involving two values of $n$, say $n_1$ and $n_2$, and find the two intensities, $I_1$ and $I_2$, that produce the same probability of detection, it follows from

$$p_n = 1 - e^{-\alpha(n^{1/\beta}I)^\beta}$$

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[1] The evolution of the result reported here is as follows. We first proved the theorem under a restriction of the function $a(n)$. Krantz then suggested that a somewhat weaker hypothesis would do, and we proved the results under his condition. Later, we read the draft of an article by Yellott (Note 1), in which he proved a related result without assuming anything about the function $a(n)$, but assuming that $p$ is strictly increasing and continuous. This brought to our attention the close connection of our problem to that of the statistics of extremes. In discussing these matters with Yellott, the question was raised whether any assumptions on $p$ were needed beyond its being a cumulative distribution function. Shortly after that, Yellott came up with a proof for his problem which, slightly adapted, served to prove the result we give here.

that

$$n_1 I_1^\beta = n_2 I_2^\beta,$$

and so

$$\beta = \frac{\log n_1/n_2}{\log I_2/I_1}.$$

For a variety of cases in vision and audition we know that doubling the duration of the stimulus—presumably rendering $n_1/n_2 = 2$—is compensated by a stimulus change of from 1.5 to 3 db (that is, $I_2/I_1$ from $2^{\frac{1}{2}}$ to 2). Thus, we estimate $\beta$ in the range from 1 to 2.

Note that it is important in estimating $\beta$ to measure intensity or energy rather than some quantity, such as pressure, which is proportional to $I^{\frac{1}{2}}$. In such cases the estimated value of the exponent will appear to be twice as large since, if $x^2 \propto I$ then $I^\beta \propto x^{2\beta}$.

## *Two Devices Displaying This Property*

Suppose a Poisson device produces counts with an intensity parameter $\mu = \alpha I^\beta$, the device is monitored for an interval of time $\Delta$, and it reports a detection if at least one count occurs within the interval. The probability of no counts is $e^{-\mu\Delta}$, so

$$\begin{aligned} P\ (\text{detection}) &= 1 - e^{-\mu\Delta} \\ &= 1 - e^{-\Delta\alpha I^\beta}. \end{aligned}$$

Such a device and the given decision rule yield the required psychometric function.

The application of such a detection model is interesting. First, it seems likely that such a device is plausible only for brief signals, since it assumes that a single count is taken as evidence for the occurrence of a signal. Second, if one envisions this process as applying when signal duration is varied, then $\Delta$ is probably not varied because it would eventually generate high false-alarm rates. Rather the outputs of a set of independent monitors are combined, and the time-intensity trade is generated. The exact form of that trade depends on $\beta$, but it is possible to predict trades that are not equal-energy contours.

As Wandell has pointed out to us, the proof of the theorem suggests a second decision mechanism yielding Equation 2 as the psychometric function. Suppose the system notes which of several channels appears to be active, calculates a statistic for each, and compares the smallest value with a criterion (i.e., selects the largest among the reciprocals of the statistics). If $n$ channels are active and $\mathbf{Y}_n$ is the smallest statistic, and if there are constants $a(n)$ such that $a(n)\mathbf{Y}_n$ has a distribution that tends to $p$, then $p$ is of the form of Equation 2.

## *Comparison with Other Psychometric Functions*

On the theoretical side, we have shown that Equation 2 is the only psychometric function having the property that if one of $n$ detectors with that psychometric function triggers a response, the resulting psychometric function is exactly parallel (on a logarithmic axis) to the elementary one. Is this property of any empirical value? Is it possible to determine whether empirical data obey this rule? The answers to such questions depend on what alternative psychometric functions are considered and the number of elementary detectors that one presumes are combined. It is impossible to give a general answer to the question, but the following relations provide some insight into the nature of the problem by showing how to compute the expected deviation for whatever alternative psychometric function one wishes to consider.

Let $p(I)$ be the psychometric function given in Equation 2. Let some other psychometric function be given by $\rho(I)$, and denote the difference between them by $\gamma(I) = p(I) - \rho(I)$. The following will be simplified if we omit the argument $I$ and denote $p(I) = p$, $\rho(I) = \rho$, and $\gamma(I) = \gamma$. Consider

$$\begin{aligned} p_n &= 1 - (1-p)^n = 1 - (1 - \rho - \gamma)^n \\ &= 1 - (1-\rho)^n + n(1-\rho)^{n-1}\gamma \\ &\qquad - n\frac{(n-1)}{2!}(1-\rho)^{n-1}(-\gamma)^2 \\ &\qquad\qquad \cdots \text{Binominal expansion}, \end{aligned}$$

$$\rho_n = 1 - (1-\rho)^n.$$

Thus,

$$\begin{aligned} |p_n - \rho_n| &\leq n(1-\rho)^{n-1}|\gamma| \\ &\qquad + \frac{n(n-1)}{2!}(1-\rho)^{n-1}\gamma^2 \cdots \\ &\leq n|\gamma| + \frac{n^2}{2!}|\gamma|^2 + \frac{n^3}{3!}|\gamma|^3 \cdots \\ &= e^{n|\gamma|} - 1. \end{aligned}$$

Although the bound may be generous, it does permit an estimate of the error. If one desires the error for the $n$ detector case to satisfy $0 < \epsilon \ll 1$, then it is sufficient that

$$e^{n|\gamma|} - 1 \leq \epsilon,$$

that is,

$$n|\gamma| \leq 1n(1+\epsilon) \cong \epsilon,$$

and so

$$|\rho - p| = |\gamma| \leq \frac{\epsilon}{n} \tag{5}$$

is sufficient.

A difficulty in applying this result is that one often does not know $n$; in fact, it is usually estimated from the data. If a situation can be arranged where $n$ is known, such as $n$ bursts of a signal or $n$ dots in a visual field, then the requirement given by Equation 5 can be tested.

*Summary*

Given an extreme decision rule, namely respond affirmatively when at least one elementary detector reports a detection, we show that the condition stated in Equation 3 yields Equation 2 as the only possible psychometric function. Some rough bounds on the parameters are given. Two mechanisms that produce such a psychometric function are presented, and the relation between this function and some other psychometric functions is derived.

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## APPENDIX

## THE FAILURE OF BRINDLEY'S ASYMPTOTIC FORMULA

Brindley (1963) claimed to show that

$$p_n = 1 - \{[1 + I + I^2/2! + \cdots + I^{m-1}/(m-1)!]e^{-I}\}^n$$

approaches, presumably uniformly,

$$1 - [e^{-I^m/m!}]^n$$

as $n$ becomes large. This is true for $m = 1$, since both equations reduce to $1 - (e^{-I})^n$. It is also true for fixed $m$ and large $I$ since $p_n \to 1$, but of more practical interest is the convergence for small values of $I$, and hence measurable $p_n$. Keeping only first-order terms in $I$, we need only compare the terms within the brackets of the two equations, for the latter,

$$e^{-I^m/m!} = 1 - I^m/m!,$$

and for the former,

$$[1+I+I^2/2!+\cdots+I^{m-1}/(m-1)!]e^{-I} \cong 1 - I^2.$$

Raising these to the $n$th power and again retaining first-order terms yields, respectively,

$$1 - nI^m/m! \quad \text{and} \quad 1 - nI^2,$$

and these are simply not the same growth in $I$, even for $m = 2$; the divergence is worse for $m = 3$ or greater.

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